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Fingering in the fast flow through porous medium

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Abstract. — The displacement of dense liquid from the pore space, e.g. by injected gas, is considered. For fast flow regimes the drag law is non-linear. The resulting nonlinear boundary problem can be solved by methods similar to dielectric-breakdown models. The gas jets form fractal structures. In the plane geometry the fractal dimension D exceeds the known linear Laplace value D_1, but it is less than $D_{0.5}$ obtained in the model with a growth parameter $\eta = 1/2$.

The problem of fluid displacement is well known. If a non-viscous liquid (e.g. water) forces a viscous one (oil) out of a pore space, then so-called viscous finger instability develops at the interface. The fingers form a fractal structure, provided the capillary effects are small enough [1-3]. Pressure is constant in the non-viscous area. In the viscous area with the linear (Darcy's) filtration law, $\mathbf{u} = -k\nabla P$, the Laplace equation $\Delta P = 0$ with fixed pressure on the moving boundary is valid. The velocity of the boundary is proportional to the gradient of pressure. The same problem arises in the simulations of the dielectric breakdown with the substitution of pressure by potential φ, and also for the cluster growth phenomena in diffusion limited aggregation [4, 5]. These processes are well studied and fractal dimensions of the resulting structures have been measured.

Non-linear problems are not so thoroughly investigated. Pietronero *et al.* [6, 7] simulated dielectric breakdown supposing the velocity of the interface to be proportional to the field to the η power. Such a dielectric-breakdown model we shall denote as DBM(η). The dependence of fractal properties on the growth parameter η was found. But this problem is described by the linear Laplace equation $\Delta\varphi = 0$, and only the boundary conditions are non-linear. The real non-linear problem is discussed briefly by Daccord *et al.* [1] who studied experimentally viscous fingering in shear-thinning fluids with $u \sim (\nabla P)^m$, where $m > 1$.

In this work we also consider an essentially non-linear problem, in which Darcy's equation is not valid. For fast filtration the square law $\rho\mathbf{u} \mid \mathbf{u} \mid /a = -b\nabla P$ is typical. Here ρ is the liquid density, a is the characteristic size of the pores, and b is a dimensionless coefficient. For

the incompressible liquid $div\ \mathbf{u} = 0$, thus the basic equation for pressure is

$$div \frac{\nabla P}{\sqrt{\mid \nabla P \mid}} = 0. \tag{1}$$

Note that instead of viscosity, the material parameter now is the liquid density. If a liquid is forced out by a low-density gas, then in the gas area the pressure gradient is small and at the interface $P \approx$ const. The velocity of the interface $u_n = -A\sqrt{\mid \nabla P \mid}$, where A is the constant coefficient. So the difference from DBM(0.5) is the non-linearity of the main equation (1).

Let us consider the initial stage of the interface instability. The form of interface is set by the equation $y = h\ \sin(kx)$, where $kh \ll 1$. The dense liquid is situated above this line. The unperturbed pressure gradient of unit value ∇P^0 is directed downwards. Taking pressure in the form $P = P^0 + P^1$, where P^1 is a small perturbation, and using a linearization, from (1) one obtains $P^1_{xx} + \frac{1}{2}P^1_{yy} = 0$. At the interface, $P^1 = -\mid \nabla P^0 \mid y = h\ \sin(kx)$. Projecting this condition onto the x axis, one gets a first-order approximation: $P^1 = h\ \sin(kx)\ \exp(-\sqrt{2}ky)$. In the linear problem the attenuation of perturbation would be slower: $\exp(-ky)$, due to the absence of the coefficient $1/2$ in the Laplace equation. In the electrostatic language the relative increase of a gradient near the crest is explained by the space charge, which partially screens charges on the surface. One can also treat (1) as the diffusion equation with a coefficient $\mathcal{D} = 1/\sqrt{\mid \nabla P \mid}$. When the gradient is large, $\mathcal{D}$ is small, leading to a larger gradient; but in the areas with low gradient, $\mathcal{D}$ is large, and gradients are short circuited there.

The velocity of the interface is $A\sqrt{\mid \nabla P \mid} = u^0(1 + \sqrt{1/2}kh\sin(kx))$. The first term is the unperturbed velocity of the interface, and the second one describes the instability with the increment $u^0k/\sqrt{2}$, i.e. $\sqrt{2}$ times slower than in the linear problem. So, for small perturbations, equation (1) does not cause qualitative differences from the linear case. But the trend of influence of non-linearity (relative slowing of growth and increase of the gradients near growing parts) is the same as for the developed instability.

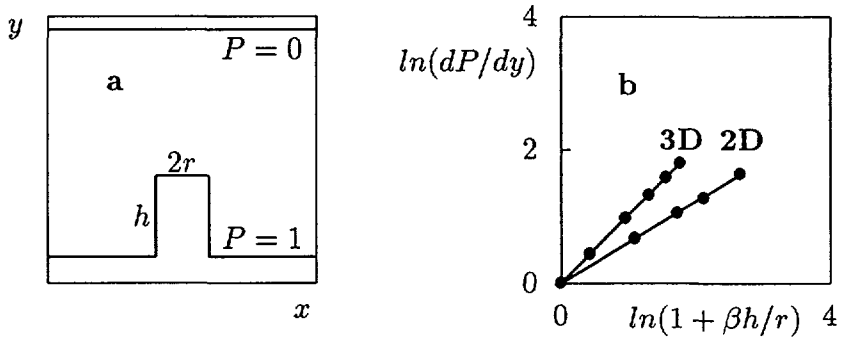

Fig. 1. — a) Rectangular finger in the pressure gradient; b) factor of field enhancement for non-linear equation. Line 2D-two-dimensional case, line 3D-three-dimensional case. The dimensionless coefficient β is 1 for 2D and 0.81 for 3D.

To consider the non-linear stage of instability, let us take a finger with height h and width $2r \ll h$ (Fig. 1a) as a model and estimate the velocity of finger growth relative to the background velocity for a flat surface. In the linear two-dimensional case the characteristic spatial

size of ∇P field near the top of the finger is $\xi \sim \sqrt{hr}$ (not r). The difference between pressures in the finger and surrounding liquid at distances more than ξ is $\nabla P^0 h$, and the pressure gradient can be estimated as $\nabla P^0 h/\xi \sim \nabla P^0 \sqrt{h/r}$. For the non-linear problem both screening and diffusion analogies show that near the top gradients will increase even higher.

Equation (1) was solved numerically in the area shown in figure 1a. For derivatives such as $\partial(\frac{\partial P/\partial x}{\sqrt{|\nabla P|}})/\partial x$ the properly centered approximation was chosen and the non-linear expression of the type:

$$P_{i,j} = f(P_{i,j}, P_{i-1,j}, P_{i-1,j-1}, P_{i-1,j+1}, P_{i+1,j}, P_{i+1,j-1}, P_{i+1,j+1}, P_{i,j-1}, P_{i,j+1})$$

was obtained. In contrast with the Laplace case, $P_{i,j}$ cannot be expressed explicity through pressure in the neighboring points. To solve this equation a method of simple iterations, naturally combined with the stabilization method, was used. The difference scheme in the three-dimensional case was built in a similar way. The results of the calculation of the non-linear problem are shown in figure 1b. For the plane case (line 2D), the experimental points are well approximated by the dependence $F = (1 + h/r)^{0.6}$ Here F is the relative enhancement of the pressure gradient taken at the center of the finger top. (For the linear problem the index 0.491, close to the theoretical value 0.5, was obtained). So, the factor of the gradient enhancement is $\sim (h/r)^{0.6}$ As expected, this factor exceeds the corresponding value for the Laplace equation. The characteristic size $\xi \sim h^{0.4} r^{0.6}$, i.e. less than in the linear case, and the top grows with the velocity $u \sim u_0 (h/r)^{0.3}$

In the three-dimensional case it seems sensible to assume that near the top of a finger the only characteristic size is the finger width, even for the non-linear problem. The results for the three-dimensional case, shown in figure 1b. (Line 3D), support this view. The best fit is: $F = (1 + 0.81h/r)^{1.01}$

So, for the growth velocity $u_n \sim \sqrt{|\nabla P|}$ the growth factor exponent in the plane case is close to 0.3, and in the three-dimensional case it is equal to 0.5. In the DBM(0.5) one has practically the same growth acceleration ($(h/r)^{0.25}$ and $(h/r)^{0.5}$) as in the non-linear case. One could expect that the fractal structure mainly depends on this growth acceleration factor. Then one might use the known data of DBM(0.5) as an estimate for the non-linear problem. Such an assumption, however, is not correct, at least in planar geometry.

The flow pattern in the initially uniform pressure gradient between parallel lines in the two-dimensional space was simulated. The model generally similar to [6, 7] was used. On the square lattice containing 60 × 60 cells the gas initially forms the bottom boundary with pressure $P = 1$. On the upper boundary $P = 0$. At each time step the gas occupies one of the cells forming a growing cluster. The conditions at vertical boundaries were periodical. Three variants of calculation were conducted:

a) non-linear model. Equation (1) was solved for every time step. The growth condition was simulated by the usual discrete rule. The probability to add one of the free perimeter sites with number i to the cluster is $\sqrt{|\nabla P_i|}/\Sigma\sqrt{|\nabla P_j|}$, where the sum is calculated over all candidates. An example of simulations is shown in figure 2a.

b) DBM(1). To compare calculations with the known results of Pietronero *et al.* [6-8], a pure linear model was examined. The Laplace equation was solved and the linear growth condition $u_n \sim \nabla P$ was used. The typical cluster is shown in figure 2b.

c) DBM(0.5): Laplace equation and non-linear growth condition. The typical pattern is shown in figure 2c.

To obtain the fractal dimension, the average density of the occupied sites in the horizontal rows $\rho(y)$ was calculated. On the diagrams ln ρ *vs.* ln y a straight line was obtained. Its slope a depends on the fractal dimension of the cluster: $a = D - 2$. The line was drawn through the lower third part of the cluster to avoid image effects in the upper boundary. The results were

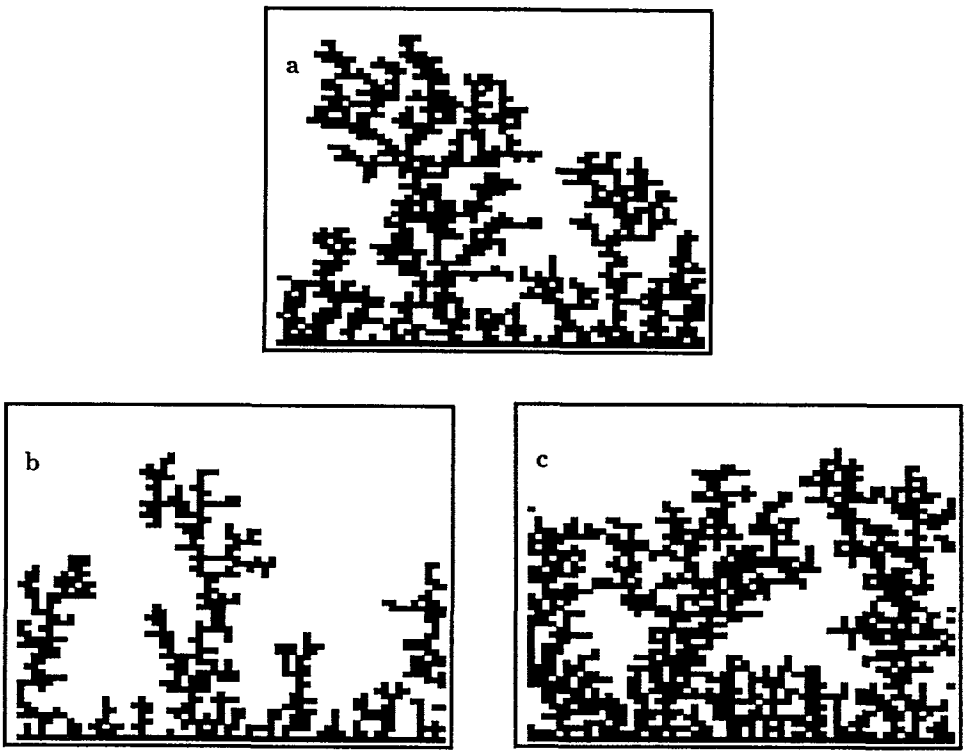

Fig. 2. — Typical flow patterns. a) non-linear case; b) linear case, DBM(1); c) linear case, but non-linear growth condition, DBM(0.5).

averaged over ~ 20 clusters. Fractal dimensions $D = 1.78$ (a), $D_1 = 1.67$ (b), $D_{0.5} = 1.88$ (c), were found with a statistical deviation of about 0.02.

The last two results are close to the data of [6-8]. Our procedure is close to Evertsz's [7] and our results agree well with his values ($D_1 = 1.64$ and $D_{0.5} = 1.89$ for the lattice width equal to 64, $D_1 = 1.663$ and $D_{0.5} = 1.918$ for the infinite lattice). The existence of the pre-determined growth direction may lead to self-affinity and consequently to different possible definitions of D (our calculations are not extensive enough to reveal deviations from self-similarity). So the agreement with the box-counting dimensions [7] may be better than one would expect. Of course, the statistical deviation does not represent all errors, and results obtained here must be considered as preliminary ones. Nevertheless, possible uncertainites are cancelled out when we regard differences between variants a) - c). So there is sufficient evidence that the non-linear dimension 1.78 is truly different from both control values.

In the case c) as compared with b) the growing clusters have dense branches which correspond to the larger fractal dimension, as in [6, 7]. The non-linear case a) produces clusters intermediate between b) and c). The difference from the linear case b) is the natural effect of the relative slower growth of the finger top.

In both cases a) and c) long fingers grow with a practically equal velocity. However in c) the field penetrates deeper in the growing cluster (less screening) and the resulting structure is somewhat denser. The distribution of the growth probability in DBM(0.5) is simply smoothed

Laplace one; DBM(η) does not change the characteristic spatial scale ξ. For the non-linear field this distribution has more contrast due to the higher gradients around the tips, and due to the mentioned effect of gradient shortening in the fiords. This explains why, with only a 20 % shift of the growth exponent - from 0.25 to 0.3, right bound being 0.5 - the fractal dimension is approximately in the middle of the corresponding interval. So, the fractal dimension depends not only on the top growth velocity, but on the whole field distribution around the finger. The transverse displacements of the interface limit the effective aspect ratio h/s. This process broadens the branches and leads to denser patterns. The difference between non-linear problem and DBM(η) was stressed also in [1].

In special experiments the porous medium was represented by a layer of 2 mm metal spheres. The liquid was forced out by air ($\sim$ 0.5 atm). A flow qualitatively similar to the calculated patterns was observed. At present the experimental results are not decisive enough to demonstrate the difference between the non-linear and viscous linear flows.

Fast displacement of the dense liquid leads to the development of jets. This phenomenon may be called "dense finger instability". The fingers branch and form a fractal structure, less ramified than in the linear case. So, this process is characterized by its own, non-trivial fractal dimension.

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